Multiple solutions to the supercritical Bahri-Coron's problem in pierced domains

Angela Pistoia, Olivier Rey

▶ **To cite this version:**

Angela Pistoia, Olivier Rey. Multiple solutions to the supercritical Bahri-Coron's problem in pierced domains. Advances in Differential Equations, 2006, 11 (6), pp.647-666. hal-00935395

HAL Id: hal-00935395

https://hal.science/hal-00935395

Submitted on 30 Jan 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Multiplicity of solutions to the supercritical Bahri-Coron's problem in pierced domains *

Angela PISTOIA[†] and Olivier REY[‡]

Abstract

We consider the supercritical Dirichlet problem

$$(P_\varepsilon) \qquad -\Delta u = u^{\frac{N+2}{N-2}+\varepsilon} \text{ in } \Omega,\ u>0 \text{ in } \Omega,\ u=0 \text{ on } \partial\Omega$$

where $N \geq 3$, $\varepsilon > 0$ and $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain with a small hole of radius d. When Ω has some symmetries, we show that (P_ε) has an arbitrary number of solutions for ε and d small enough. When Ω has no symmetries, we prove the existence, for d small enough, of solutions blowing up at two or three points close to the hole as ε goes to zero.

1 Introduction

We consider the problem

$$-\Delta u = u^q \text{ in } \Omega,\ u>0 \text{ in } \Omega,\ u=0 \text{ on } \partial\Omega, \tag{1.1}$$

where $N \geq 3$, $q > 1$ and Ω is a smooth and bounded domain in $\mathbb{R}^N$.

In the subcritical case, i.e. $q < \frac{N+2}{N-2}$, problem (1.1) has a solution for any domain Ω. In the critical case, i.e. $q = \frac{N+2}{N-2}$, Pohozaev's identity (see [20]) shows that (1.1) has no solution when Ω is starshaped, whereas Kazdan and

*The first author is supported by M.U.R.S.T., project "Metodi variazionali e topologici nello studio di fenomeni non lineari".

†Dipartimento di Metodi e Modelli Matematici, Università di Roma "La Sapienza", 00100 Roma, Italy.

‡Centre de Mathématiques Laurent Schwartz, Ecole Polytechnique, 91128 Palaiseau Cedex, France.

Warner [12] proved the existence of a radial solution on annular domains. Later, Coron [7] proved that (1.1) has a solution when Ω has a small hole. Such a result was greatly extended by Bahri and Coron [3], showing that the nontriviality of some homology group of Ω ensures the existence of a solution. However, this sufficient topological condition is not necessary for solvability of (1.1), as examples show ([8], [11], [18]). In the supercritical case, i.e. $q > \frac{N+2}{N-2}$, the condition appears to be neither necessary [14], nor sufficient, at least as $q > \frac{N+1}{N-3}$ [19].

The present paper is concerned with the slightly supercritical case, $q = \frac{N+2}{N-2} + \varepsilon$, where ε is a small positive parameter. We are interested in finding solutions to (1.1) which blow up at some points of Ω as ε goes to zero, in the following sense. We consider, for $\lambda > 0$ and $\xi \in \mathbb{R}^N$, the functions

$$U_{\lambda,\xi}(x) = \alpha_N \frac{\lambda^{\frac{N-2}{2}}}{(\lambda^2 + |x-\xi|^2)^{\frac{N-2}{2}}} \quad x \in \mathbb{R}^N, \ \alpha_N = [N(N-2)]^{\frac{N-2}{4}}$$

which are the only positive solutions to the equation $-\Delta U = U^{\frac{N+2}{N-2}}$ in $\mathbb{R}^N$ ([1], [6], [23]) and the projections $P_\Omega U_{\lambda,\xi}$ of $U_{\lambda,\xi}$ onto $\mathrm{H}^1_0(\Omega)$, i.e.

$$\Delta PU_{\lambda,\xi} = \Delta U_{\lambda,\xi} \text{ in } \Omega, \ PU_{\lambda,\xi} = 0 \text{ on } \partial\Omega.$$

These functions are, as λ goes to zero, approximate solutions to (1.1) in the critical case. Denoting (P_ε) problem (1.1) with $q = \frac{N+2}{N-2} + \varepsilon$, we set:

Definition 1.1. *Let $k \in \mathbb{N}^*$ and u_ε be a solution to (P_ε). We say that $(u_\varepsilon)_\varepsilon$ blows up at k points $\xi_1, \dots, \xi_k$ of Ω, $\xi_i \neq \xi_j$ for $i \neq j$, as ε goes to zero, if and only if there exist positive parameters $\lambda_{1\varepsilon}, \dots, \lambda_{k\varepsilon}$ and points $\xi_{1\varepsilon}, \dots, \xi_{k\varepsilon}$ in Ω such that*

$$u_\varepsilon - \sum_{i=1}^{k} PU_{\lambda_{i\varepsilon},\xi_{i\varepsilon}} \to 0 \text{ as } \varepsilon \to 0$$

in $\mathrm{H}^1_0(\Omega)$ and $C^1\big(\Omega \setminus \bigcup_{i=1}^{k} B(\xi_i, a)\big)$, for any $a > 0$.

In the slightly subcritical case, i.e. $q = \frac{N+2}{N-2} - \varepsilon$, it was proved that such solutions do exist [4]. In particular it is always true, whatever Ω may be, for $k = 1$. (Some examples of domains with solutions blowing up at several points are given in [17].) When $q = \frac{N+2}{N-2} + \varepsilon$, the situation turns out to be different. In particular, solutions blowing up at a single point never exist [5]. However, Del Pino, Felmer and Musso [9] proved the existence of solutions blowing up at two points, provided that Ω satisfies some topological assumption - for example, Ω has a small hole. (See also [13] for a simplified assumption when $N = 3$.) Existence of a large number of solutions to (P_ε) was found when Ω is a symmetric annulus [10] or a symmetric annulus with a "channel" [14] (making the domain contractible): for such domains Ω, there exists an integer $k(\Omega)$ such that for any $k \geq k(\Omega)$, problem (P_ε) has, for ε small enough, a solution which blows up at k points as ε goes to zero. (We note that k may need to be chosen very large to obtain a $k-$peaked solution.) The aim of this paper is to resume the study of the problem in order to obtain existence and multiplicity results which improve the previous ones, both in the symmetric and the nonsymmetric cases.

In Section 2, we consider a symmetric domain with a small hole. Namely, writing $x \in \mathbb{R}^N$ as $x = (x', x'')$, with $x' \in \mathbb{R}^2$ and $x'' \in \mathbb{R}^{N-2}$, we assume that

(i) Ω is rotationally invariant with respect to x', i.e. if $(x', x'') \in \Omega$, then $(Ax', x'') \in \Omega$ for any central rotation A of the $x'-$plane.

(ii) Ω is symmetric with respect to x_j, $3 \leq j \leq N$, i.e., for any $j = 3, \ldots, N$, $(x', x_3, \ldots, x_j, \ldots, x_N) \in \Omega$ implies that $(x', x_3, \ldots, -x_j, \ldots, x_N) \in \Omega$.

(iii) Ω contains a ball centered in zero, i.e. there exists $a > 0$ such that $B(0, a) \subset \Omega$.

For $\delta \in (0, a)$, we set

$$\Omega_\delta = \{x \in \Omega \ : \ |x| > \delta\}$$

and we consider (P_ε) with $\Omega = \Omega_\delta$. We prove:

Theorem 1.1. *Let $\Omega = \Omega_\delta$. For any $k \geq 2$, there exists $\delta_k > 0$ such that for any $\delta \in (0, \delta_k)$, (P_ε) has, for ε small enough, a solution which blows up at k points in Ω_δ as ε goes to zero.*

Corollary 1.1. *Let $\Omega = \Omega_\delta$. For any $h \geq 1$, there exists $\delta_h > 0$ such that for any $\delta \in (0, \delta_h)$, (P_ε) has, for ε small enough, h (rotationally) distinct solutions which blow up at two, three, $\ldots$, $h+1$ points respectively as ε goes to zero.*

In the critical case, a small hole generates the existence of a solution to (1.1) - see [7], [22]. As such a solution is topologically nontrivial, i.e. induces some difference of topology between the level sets of a functional associated to the problem, this solution continues for ε small enough. This provides us with an additional solution to (P_ε) on Ω_δ, which goes to a solution to (P_0) as ε goes to zero.

In Section 3, we consider a domain with both a small hole and a small "channel", making it contractible. Namely, for $0 < \delta < R$ and $\sigma > 0$ we set

$$\Omega_{\delta,\sigma} = \left\{ x \in \mathbb{R}^N \ : \ \delta < |x| < R, \ \left(\sum_{i=1}^{N-1} x_i^2 \right)^{1/2} > \sigma x_N \right\}$$

and similarly to Theorem 1.1 we prove:

Theorem 1.2. *Let $\Omega = \Omega_{\delta,\sigma}$. For any $k \geq 2$, there exist $\delta_k > 0$, $\sigma_k > 0$, such that for any $\delta \in (0, \delta_k)$ and $\sigma \in (0, \sigma_k)$, (P_ε) has, for ε small enough, a solution which blows up at k points in $\Omega_{\delta,\sigma}$ as ε goes to zero.*

Corollary 1.2. *Let $\Omega = \Omega_{\delta,\sigma}$. For any $h \geq 1$, there exist $\delta_h > 0$, $\sigma_h > 0$, such that for any $\delta \in (0, \delta_h)$ and $\sigma \in (0, \sigma_h)$, (P_ε) has, for ε small enough, h (rotationally) distinct solutions which blow up at two, three, $\ldots$, $h+1$ points respectively as ε goes to zero.*

In Section 4, we drop any symmetry assumption and just consider a domain with a small hole. Namely, Ω being a smooth bounded domain in $\mathbb{R}^N$, we may assume, up to a translation, that Ω contains some ball $B(0, r)$, $r > 0$. Then, for $d \in (0, r)$, we set

$$\Omega_d = \{ x \in \Omega \ : \ |x| > d \}$$

and we first look for a solution blowing up at two points as ε goes to zero. As already stated, existence of such a solution has been proved in [9]. We obtain additional informations:

Theorem 1.3. *Let $\Omega = \Omega_d$. There exists $d_0 > 0$ such that for any $d \in (0, d_0)$, (P_ε) has, for ε small enough, a solution which blows up at two points ξ_1 and ξ_2 in Ω_d as ε goes to zero. Moreover $|\xi_i| \sim r_0 d$ and $\xi_2 = -\xi_1 + o(d)$ as ε goes to zero, where $r_0 > 1$ is the unique solution in $(1, \infty)$ of*

$$\frac{1}{2^{N-1} r_0^N} = \frac{1}{(r_0^2 - 1)^{N-1}} + \frac{1}{(r_0^2 + 1)^{N-1}}.$$

We also look for solutions blowing up at three points, and we obtain a result of the same kind - see Theorem 4.4. Such a result is of qualitative importance, as it shows that the two peaks solutions are not the only blowing up solutions existing. Whereas single peak solutions do not exist, solutions with three peaks or more may be exhibited, without any symmetry assumptions on the domain (the method that we use extends to the study of solutions blowing up at k points, for any $k \geq 2$). Such kind of results was expected, but had never been proved because of technical obstacles.

Actually, using a rescaling, we obtain that the nonsymmetric case is asymptotically equivalent, as the hole radius goes to zero, to the symmetric one. Then, we are left with proving that peaked solutions in the symmetric case are stable through C^1-perturbations. We study this general stability, and check that it is satisfied in the cases we are interested in.

Before turning to the proofs of the results, we introduce some notation. A domain Ω being given, we denote by G_Ω the Green's function of the negative Laplacian in Ω with Dirichlet boundary conditions, i.e.

$$-\Delta G_\Omega(\cdot,\xi) = a_N \delta_\xi \text{ in } \Omega,\ G_\Omega(\cdot,\xi) = 0 \text{ in } \partial\Omega$$

with $a_N = \big((N-2)\text{meas}(S^{N-1})\big)^{-1}$, S^{N-1} being the $(N-1)-$dimensional unit sphere, and by H_Ω its regular part, i.e.

$$H_\Omega(x,\xi) = \frac{1}{|x-\xi|^{N-2}} - G_\Omega(x,\xi),\ (x,\xi) \in \Omega\times\Omega.$$

We also define the Robin function as

$$R_\Omega(x) = H(x,x), \quad x \in \Omega.$$

We point out that, as a consequence of the maximum principle

$$P_\Omega U_{\lambda,\xi}(x) = U_{\lambda,\xi}(x) - \alpha_N \lambda^{\frac{N-2}{2}} H_\Omega(x,\xi) + O\left(\lambda^{\frac{N+2}{2}} \left(\text{dist } (\xi,\partial\Omega)\right)^{-N}\right).$$

Such an expansion introduces us to the role that Green's function and its regular part play in that kind of problems (see e.g. [2], [21]), and in the following arguments.

2 Proof of Theorem 1.1

According to assumptions (i), (ii), (iii) in the previous section, let $k \geq 2$ be fixed and $R > 0$ such that $(x', 0'') \in \Omega_\delta$ if and only if $\delta < |x'| < R$.

Step 1 $-$ Reduction of the problem to a finite dimensional one.

Problem (1.1) has a variational structure. Indeed, defining the functional

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{q+1}\int_\Omega (u_+)^{q+1}, \quad u \in \mathrm{H}^1_0(\Omega) \cap \mathrm{L}^{q+1}(\Omega)$$

the strong maximum principle ensures that the critical points $u \not\equiv 0$ of J are solutions of the problem. The method initially developed in [2], [21] for $q = \frac{N+2}{N-2}$, consists in looking for solutions written as

$$u_\varepsilon = \alpha_N \sum_{i=1}^{k} P_\Omega U_{\lambda_i,\xi_i} + v_\varepsilon$$

where v_ε is orthogonal for the H_0^1-scalar product to the derivatives of $P_\Omega U_{\lambda_i,\xi_i}$ with respect to the parameters λ_i, ξ_i, and assumed to be small in H_0^1-norm. (It is shown that $\lambda = (\lambda_1, \dots, \lambda_k)$, $\xi = (\xi_1, \dots, \xi_k)$ and v_ε build a good local parametrization of $\mathrm{H}_0^1(\Omega)$.) Then, once $v_\varepsilon(\lambda, \xi)$ is found such that $\frac{\partial J}{\partial v}\big(\lambda, \xi, v_\varepsilon(\lambda, \xi)\big) = 0$, the initial problem reduces to a finite dimensional one involving λ and ξ parameters only.

Such a method was adapted in [9], using weighted Hölder spaces, to the supercritical case $q = \frac{N+2}{N-2} + \varepsilon$. According to [9] the problem reduces, for ε small enough, to finding critical points of a finite dimensional functional written as

$$\widetilde{J}(\Lambda, \xi) = \frac{1}{2}\Big(\sum_{i=1}^{k} R_\Omega(\xi_i)\Lambda_i^2 - \sum_{\substack{i,j=1 \\ i\neq j}}^{k} G_\Omega(\xi_i, \xi_j)\Lambda_i\Lambda_j\Big) + \sum_{i=1}^{k} \ln \Lambda_i + \varphi_\varepsilon(\Lambda, \xi)$$

where $\lambda = c_N \varepsilon^{\frac{1}{N-2}} \Lambda$ (c_N is a positive constant depending on N only), and φ_ε goes to zero in $\mathrm{C}^1_{loc}\big((\mathbb{R}_+^*)^k \times (\Omega_\delta^k \setminus D)\big)$ as ε goes to zero, with $D = \big\{\xi \in (\mathbb{R}^N)^k \ : \ \xi_i = \xi_j \text{ for some } i \neq j\big\}$. As a consequence, defining

$$\Psi(\Lambda, \xi) = \frac{1}{2}\Big(\sum_{i=1}^{k} R_\Omega(\xi_i)\Lambda_i^2 - \sum_{\substack{i,j=1 \\ i\neq j}}^{k} G_\Omega(\xi_i, \xi_j)\Lambda_i\Lambda_j\Big) + \sum_{i=1}^{k} \ln \Lambda_i$$

stable critical points of Ψ (i.e. critical points which are stable under small C^1-perturbations) such that $\xi_i \neq \xi_j$ for $i \neq j$, provide us with solutions to (P_ε). Now, taking also into account the symmetries of the domain Ω_δ, we can look, as in [10], [14], for critical points (Λ, ξ) of Ψ such that

$$\Lambda_i = \Lambda, \ \xi_i(\rho) = \Big(\rho \cos \frac{2\pi(i-1)}{k}, \rho \sin \frac{2\pi(i-1)}{k}, 0''\Big), \ 1 \leq i \leq k \tag{2.2}$$

with $\rho \in (\delta, R)$. Then, we are left with the two variables function ψ_δ : $\mathbb{R}_+^* \times (\delta, R) \to \mathbb{R}$ defined as

$$\psi_\delta(\Lambda, \rho) = k\left(\frac{\Lambda^2}{2}\gamma_\delta(\rho) + \ln \Lambda\right)$$

with

$$\gamma_\delta(\rho) = R_{\Omega_\delta}(\xi_1(\rho)) - \sum_{i=2}^{k} G_{\Omega_\delta}(\xi_1(\rho), \xi_i(\rho)).$$

<u>Step 2</u> – <u>For δ small enough, ψ_δ has a stable critical point.</u>

This will conclude the proof of Theorem 1.1. We perform the change of variables

$$r = \frac{\rho}{\delta}, \quad \mu = \frac{\Lambda}{\delta^{\frac{N-2}{2}}}$$

and we define the function $\phi_\delta : \mathbb{R}_+^* \times (1, R/\delta) \to \mathbb{R}$ as

$$\phi_\delta(\mu, r) = \psi_\delta(\Lambda, \rho) - k\frac{N-2}{2}\ln\delta = k\left(\frac{\mu^2}{2}\Gamma_\delta(r) + \ln\mu\right)$$

with

$$\Gamma_\delta(r) = \delta^{N-2}\gamma_\delta(\delta r) = R_{\Omega_\delta/\delta}(\xi_1(r)) - \sum_{i=2}^{k} G_{\Omega_\delta/\delta}(\xi_1(r), \xi_i(r)).$$

Obviously, (μ, r) is a stable critical point of ϕ_δ if and only if $(\delta^{\frac{N-2}{2}}\mu, \delta r)$ is a stable critical point of ψ_δ. Setting

$$E = \left\{x \in \mathbb{R}^N \ : \ |x| > 1\right\}$$

we remark that $\Omega_\delta/\delta \subset E$ and Ω_δ/δ goes to E as δ goes to zero, i.e. for any compact set $K \subset E$, there exists $\delta_K > 0$ such that for any $\delta \in (0, \delta_K)$, $K \subset \Omega_\delta/\delta$. Moreover, it is easily checked, using the maximum principle, that $R_{\Omega_\delta/\delta}(x) = H_{\Omega_\delta/\delta}(x,x)$ goes to $R_E(x) = H_E(x,x)$ in $\mathrm{C}^1_{loc}(E)$ and $G_{\Omega_\delta/\delta}(x,y)$ goes to $G_E(x,y)$ in $\mathrm{C}^1_{loc}(E^2 \setminus \Delta)$ as δ goes to zero, where Δ is the diagonal of E^2, i.e. $\Delta = \{(x,y) \in E^2 \ : \ x = y\}$. Therefore, ϕ_δ goes to ϕ_E in $\mathrm{C}^1_{loc}(\mathbb{R}_+^* \times (1,\infty))$, where $\phi_E : \mathbb{R}_+^* \times (1,\infty) \to \mathbb{R}$ is defined as

$$\phi_E(\mu, r) = k\left(\frac{\mu^2}{2}\Gamma_E(r) + \ln\mu\right)$$

with

$$\Gamma_E(r) = R_E(\xi_1(r)) - \sum_{i=2}^{k} G_E(\xi_1(r), \xi_i(r)).$$

Then, the only thing that remains to be proved is that ϕ_E has a stable critical point. We have

$$R_E(\xi) = \frac{1}{(|\xi|^2 - 1)^{N-2}}, \ G_E(\xi, \zeta) = \frac{1}{|\xi - \zeta|^{N-2}} - \frac{1}{\left||\zeta|\xi - \frac{\zeta}{|\zeta|}\right|^{N-2}}. \quad (2.3)$$

Consequently

$$\begin{aligned}\Gamma_E(r) =& \frac{1}{(r^2-1)^{N-2}} - \sum_{i=1}^{k-1} \frac{1}{2^{\frac{N-2}{2}} r^{N-2} \left(1-\cos\frac{2\pi i}{k}\right)^{\frac{N-2}{2}}} \\ &+ \sum_{i=1}^{k-1} \frac{1}{\left(r^4+1-2r^2\cos\frac{2\pi i}{k}\right)^{\frac{N-2}{2}}}.\end{aligned} \tag{2.4}$$

Clearly, $\lim_{r\to 1^+} \Gamma_E(r) = +\infty$ and $\lim_{r\to+\infty} r^{N-2}\Gamma_E(r) = l < 0$. Moreover, the cancellation of $\Gamma_E(r)$ implies that $\Gamma'_E(r) < 0$. Indeed

$$\begin{aligned}\Gamma'_E(r) =& -(N-2)\left[\frac{2r}{(r^2-1)^{N-1}} - \sum_{i=1}^{k-1} \frac{1}{2^{\frac{N-2}{2}} r^{N-1} \left(1-\cos\frac{2\pi i}{k}\right)^{\frac{N-2}{2}}}\right. \\ &\left. + \sum_{i=1}^{k-1} \frac{2r\left(r^2-\cos\frac{2\pi}{k}i\right)}{\left(r^4+1-2r^2\cos\frac{2\pi i}{k}\right)^{\frac{N}{2}}}\right]\end{aligned} \tag{2.5}$$

whence, in view of (2.4), if $\Gamma_E(r) = 0$

$$\Gamma'_E(r) = -(N-2)\left[\frac{r^2+1}{r(r^2-1)^{N-1}} + \sum_{i=1}^{k-1} \frac{r^4-1}{r\left(r^4+1-2r^2\cos\frac{2\pi i}{k}\right)^{\frac{N}{2}}}\right] < 0.$$

Then, there exists a unique $r^* \in (1,\infty)$ such that $\Gamma_E(r^*) = 0$. $\Gamma_E(r) < 0$ for $r > r^*$, and there is some $r_0 > r^*$ which is a global minimum of ϕ_E in $(1,\infty)$ and a isolated critical point because of analyticity. On the other hand, for any $r > r^*$, there exists a unique $\mu = \mu(r) = (-\Gamma_E(r))^{-1/2}$ such that $\frac{\partial \phi_E}{\partial \mu}(\mu(r), r) = 0$. Moreover, $\frac{\partial^2 \phi_E}{\partial \mu \partial r}(\mu(r), r) = 0$ and $\frac{\partial^2 \phi_E}{\partial \mu^2}(\mu(r), r) = 2k\Gamma_E(r) < 0$. Therefore, a standard linking argument shows that $(\mu(r_0), r_0)$ is a critical point of ϕ_E which is stable under C^1-perturbations. This concludes the proof of Theorem 1.1.

3 Proof of Theorem 1.2

Let $k \geq 2$ be fixed and $R > 0$ such that $(x', 0'') \in \Omega_{\delta,\sigma}$ if and only if $\delta < |x'| < R$.

<u>Step 1</u> – <u>Reduction of the problem to a finite dimensional one.</u>

We argue exactly as in the previous section. The only change is that the symmetry with respect to x_N being lost, we have to set, instead of (2.2)

$$\xi(\rho,\tau)=\Big(\rho\cos\frac{2\pi(i-1)}{k},\rho\sin\frac{2\pi(i-1)}{k},0,\dots,0,\tau\Big),\ 1\le i\le k \qquad (3.6)$$

with

$$(\rho,\tau)\in S_{\delta,\sigma}=\left\{(\rho,\tau)\in\mathbb{R}_+^*\times\mathbb{R}\ :\ \delta^2<\rho^2+\tau^2<R,\ \rho>\sigma\tau\right\}.$$

Then we have to consider, instead of ψ_δ, a three variables function $\psi_{\delta,\sigma}$: $\mathbb{R}_+^*\times S_{\delta,\sigma}\to\mathbb{R}$ defined as

$$\psi_{\delta,\sigma}(\Lambda,\rho,\tau)=k\left(\frac{\Lambda^2}{2}\gamma_{\delta,\sigma}(\rho,\tau)+\ln\Lambda\right)$$

and

$$\gamma_{\delta,\sigma}(\rho,\tau)=R_{\Omega_{\delta,\sigma}}(\xi_1(\rho,\tau))-\sum_{i=2}^{k}G_{\Omega_{\delta,\sigma}}(\xi_1(\rho,\tau),\xi_i(\rho,\tau)).$$

Step 2 – For δ and σ small enough, $\psi_{\delta,\sigma}$ has a stable critical point.

We perform, as previously, a change of variables

$$r=\frac{\rho}{\delta},\quad t=\frac{\tau}{\delta},\quad \mu=\frac{\Lambda}{\delta^{\frac{N-2}{2}}}$$

and we define the function $\phi_{\delta,\sigma}:\Sigma_{\delta,\sigma}\to\mathbb{R}$ as

$$\phi_{\delta,\sigma}(\mu,r,t)=\psi_{\delta,\sigma}(\Lambda,\rho,\tau)-k\frac{N-2}{2}\ln\delta=k\left(\frac{\mu^2}{2}\Gamma_{\delta,\sigma}(r,t)+\ln\mu\right)$$

with $\Gamma_{\delta,\sigma}(r,t)=\delta^{N-2}\gamma_{\delta,\sigma}(\delta r,\delta t)$, i.e.

$$\Gamma_{\delta,\sigma}(r,t)=R_{\Omega_{\delta,\sigma}/\delta}(\xi_1(r,t))-\sum_{i=2}^{k}G_{\Omega_{\delta,\sigma}/\delta}(\xi_1(r,t),\xi_i(r,t))$$

and

$$\Sigma_{\delta,\sigma}=\left\{(r,t)\in\mathbb{R}_+^*\times\mathbb{R}\ :\ 1<r^2+t^2<\frac{R}{\delta^2},\ r>\sigma t\right\}.$$

Of course (μ,r,t) is a stable critical point of $\phi_{\delta,\sigma}$ if and only if $\left(\delta^{\frac{N-2}{2}}\mu,\delta r,\delta t\right)$ is a stable critical point of $\psi_{\delta,\sigma}$. Setting

$$E^*=E\setminus\{(0,\dots,0,x_N)\ :\ x_N>1\},\ E=\left\{x\in\mathbb{R}^N\ :\ |x|>1\right\}$$

we remark that $\Omega_{\delta,\sigma}/\delta \subset E^*$ and $\Omega_{\delta,\sigma}/\delta$ goes to E^* as δ and σ go to zero, i.e. for any compact set $K \subset E^*$, there exist $\delta_K > 0$, $\sigma_K > 0$ such that for any $\delta \in (0, \delta_K)$ and $\sigma \in (0, \sigma_K)$, $K \subset \Omega_{\delta,\sigma}/\delta$. Moreover, it is easily checked, using the maximum principle, that $H_{\Omega_{\delta,\sigma}/\delta}(x,x)$ goes to $H_E(x,x)$ in $C^1_{loc}(E^*)$ and $G_{\Omega_{\delta,\sigma}/\delta}(x,y)$ goes to $G_E(x,y)$ in $C^1_{loc}((E^*)^2 \setminus \Delta)$ as δ and σ go to zero, where Δ is the diagonal of $(E^*)^2$. Consequently, $\phi_{\delta,\sigma}$ goes to ϕ_{E^*} in $C^1_{loc}(\mathbb{R}^*_+ \times \Sigma_0)$, where $\phi_{E^*} : \mathbb{R}^*_+ \times \Sigma_0 \to \mathbb{R}$ is defined as

$$\phi_{E^*}(\mu, r, t) = k\left(\frac{\mu^2}{2}\Gamma_{E^*}(r,t) + \ln \mu\right)$$

with

$$\Gamma_{E^*}(r,t) = R_E(\xi_1(r,t)) - \sum_{i=2}^{k} G_E(\xi_1(r,t), \xi_i(r,t)) \tag{3.7}$$

and

$$\Sigma_0 = \left\{(r,t) \in \mathbb{R}^*_+ \times \mathbb{R} \ : \ 1 < r^2 + t^2\right\} \setminus \{(0,t) \ : \ t > 1\}.$$

In view of the previous arguments, Theorem 1.2 will follow from the existence of a stable critical point of ϕ_{E^*}. From (2.3), (3.6) and (3.7), we deduce that

$$\begin{aligned}\Gamma_{E^*}(r,t) =& \frac{1}{(r^2+t^2-1)^{N-2}} - \sum_{i=1}^{k-1} \frac{1}{2^{\frac{N-2}{2}} r^{N-2} \left(1 - \cos\frac{2\pi i}{k}\right)^{\frac{N-2}{2}}} \\ &+ \sum_{i=1}^{k-1} \frac{1}{\left((r^2+t^2)^2 + 1 - 2r^2\cos\frac{2\pi i}{k} - 2t^2\right)^{\frac{N-2}{2}}}.\end{aligned}$$

We remark that for any $r > 1$, there exists a unique $t(r)$ such that $\frac{\partial \Gamma_{E^*}}{\partial t}(r, t(r)) = 0$, i.e. $t(r) = 0$. Moreover, $\Gamma_{E^*}(r,0) = \max_{t\in\mathbb{R}} \Gamma_{E^*}(r,t)$, and straightforward computations yield $\frac{\partial^2 \Gamma_{E^*}}{\partial t^2}(r,0) < 0$, $\frac{\partial^2 \Gamma_{E^*}}{\partial t \partial r}(r,0) = 0$. We note that $\Gamma_{E^*}(r,0)$ coincides with $\Gamma_E(r)$ in the previous section. Then, let r_0 be a global minimum of $\Gamma_{E^*}(r,0)$ for $r \in (1,\infty)$, and $\mu(r_0) = (-\Gamma_{E^*}(r_0))^{-1/2}$. $(\mu(r_0), r_0, 0)$ is a critical point of ϕ_{E^*}, and

$$\phi''_{E^*}(\mu(r_0), r_0, 0) = k\begin{pmatrix} 2\Gamma_E(r_0) & 0 & 0 \\ 0 & \frac{(\mu(r_0))^2}{2}\Gamma''_E(r_0) & 0 \\ 0 & 0 & \frac{(\mu(r_0))^2}{2}\frac{\partial^2 \Gamma_{E^*}}{\partial t^2}(r_0,0) \end{pmatrix}.$$

We know that $\Gamma_E(r_0) < 0$, $\frac{\partial^2 \Gamma_{E^*}}{\partial t^2}(r_0,0) < 0$, and $\Gamma''_E(r_0) \geq 0$. Even if $\Gamma''_E(r_0) = 0$, the fact that r_0 is a minimum and an isolated critical point of $\Gamma_E(r)$ (and of $r \mapsto \phi_{E^*}(\mu(r_0), r, 0)$ as well) would still ensure, through a standard linking argument, that $(\mu(r_0), r_0, 0)$, as a critical point of ϕ_{E^*}, is stable under C^1-perturbations. This concludes the proof of Theorem 1.2.

4 The nonsymmetric case

The general argument.

In this last section, we drop the symmetry assumptions on the domain. Ω being any smooth bounded domain in $\mathbb{R}^N$, that we may assume, up to a translation, contains a ball $B(0, R)$ for some $R > 0$, we set, for $0 < d < R$

$$\Omega_d = \{x \in \Omega \ : \ |x| > d\}$$

i.e. $\Omega_d = \Omega \setminus \overline{B(0,d)}$. We consider (P_ε) in Ω_d and, as previously, we look for a solution blowing up at k points as ε goes to zero, $k \geq 2$. According to Section 1, we know that the problem may be reduced to finding, for ε small enough, a critical point of

$$\widetilde{J}(\Lambda, \xi) = \frac{1}{2}\Big(\sum_{i=1}^{k} R_{\Omega_d}(\xi_i)\Lambda_i^2 - \sum_{\substack{i,j=1\\ i\neq j}}^{k} G_{\Omega_d}(\xi_i, \xi_j)\Lambda_i\Lambda_j\Big) + \sum_{i=1}^{k} \ln \Lambda_i + \varphi_{\Omega_d,\varepsilon}(\Lambda, \xi)$$

in $\Sigma = (\mathbb{R}_+^*)^k \times (\Omega_d^k \setminus D)$, with $D = \{\xi \in (\mathbb{R}^N)^k \ : \ \xi_i = \xi_j \text{ for some } i \neq j\}$, $\varphi_{\Omega_d,\varepsilon}(\Lambda, \xi)$ going to zero in $C^1_{loc}(\Sigma)$. We consider, as previously, the main part in $\widetilde{J}(\Lambda, \xi)$, that is, we set

$$\psi(\Lambda, \xi) = \frac{1}{2}\Big(\sum_{i=1}^{k} R_{\Omega_d}(\xi_i)\Lambda_i^2 - \sum_{\substack{i,j=1\\ i\neq j}}^{k} G_{\Omega_d}(\xi_i, \xi_j)\Lambda_i\Lambda_j\Big) + \sum_{i=1}^{k} \ln \Lambda_i$$

for $(\Lambda, \xi) \in \Sigma$. Through the rescaling

$$x_i = \frac{\xi_i}{d}, \qquad \mu_i = \frac{\Lambda_i}{d^{\frac{N-2}{2}}}$$

we have

$$\psi(\Lambda, \xi) = \phi(\mu, x) + k\frac{N-2}{2}\ln d$$

with

$$\phi(\mu, x) = \frac{1}{2}\Big(\sum_{i=1}^{k} R_{\Omega_d/d}(x_i)\mu_i^2 - \sum_{\substack{i,j=1\\ i\neq j}}^{k} G_{\Omega_d/d}(x_i, x_j)\mu_i\mu_j\Big) + \sum_{i=1}^{k} \ln \mu_i$$

as $(\mu, x) \in (\mathbb{R}_+^*)^k \times ((\Omega_d/d)^k \setminus D)$. We know that Ω_d/d goes to $E = \{x \in \mathbb{R}^N \ : \ |x| > 1\}$, $H_{\Omega_d/d}$ goes to H_E in $\mathrm{C}^1_{loc}(E)$ and $G_{\Omega_d/d}$ goes to G_E in $\mathrm{C}^1_{loc}(E^2 \setminus \Delta)$ as d goes to zero (we recall that Δ is the diagonal of E^2).

Let us consider the limit function

$$\phi_E(\mu, x) = \frac{1}{2}\Big(\sum_{i=1}^{k} R_E(x_i)\mu_i^2 - \sum_{\substack{i,j=1 \\ i \neq j}}^{k} G_E(x_i, x_j)\mu_i\mu_j\Big) + \sum_{i=1}^{k} \ln \mu_i. \qquad (4.8)$$

As we saw, this function has a critical point (μ^*, x^*) which may be written as

$$\begin{aligned} \mu_i^* &= (-\Gamma_E(r_0))^{-1/2} \\ x_i^* &= \left(r_0 \cos \frac{2\pi(i-1)}{k}, r_0 \sin \frac{2\pi(i-1)}{k}, 0, \ldots, 0\right), \ 1 \leq i \leq k \end{aligned} \qquad (4.9)$$

where r_0 is a global minimizer of $\Gamma_E(r)$ in $(1, \infty)$. Of course, this critical point is degenerate, since ϕ_E is invariant under central rotations of E^k. Let $\mathcal{M}$ be the manifold of critical points generated by these rotations, i.e.

$$\mathcal{M} = \{(\mu^*, \mathcal{R}x^*) \ : \ \mathcal{R} \text{ is a central rotation of } E^k\}.$$

Let us assume that $\phi_E''(\mu^*, x^*)$ is nondegenerate in the orthogonal subspace to $\mathrm{T}_{(\mu^*,x^*)}\mathcal{M}$ (the same is true at any $(\mu, x) \in \mathcal{M}$). Then, a standard linking argument shows that a C^1-perturbation of ϕ_E has a critical point in a neighborhood of $\mathcal{M}$. Whence, coming back to $\widetilde{J}$, the existence of $d_0 > 0$ such that for any $0 < d < d_0$, there exists some $\varepsilon_d > 0$ such that, for any $0 < \varepsilon < \varepsilon_d$, $\widetilde{J}$ has a critical point $(\Lambda, \xi) = \left(d^{\frac{N-2}{2}}\mu, dx\right)$, with the distance of (μ, x) to $\mathcal{M}$ going to zero as d and ε go to zero.

We remark that dim $\mathcal{M} = N - 1$ as $k = 2$ ($x^* = (x_1^*, -x_1^*)$ and x_1^* runs on a $(N-1)-$sphere). As $k \geq 3$, the position of $\mathcal{R}x^*$ is correctly defined by the orientation of the plane in which the x_i^*'s are, and by a central rotation in that plane. Therefore, for $k \geq 3$, dim $\mathcal{M} = 2(N-2) + 1 = 2N - 3$ ($2(N-2)$ is the dimension of the $2-$Grassmanian in $\mathbb{R}^N$). Finally, we can state the following general result:

Proposition 4.1. *Let $k \in \mathbb{N}$, $k \geq 2$, ϕ_E and (μ^*, x^*) defined by (4.8) and (4.9). If*

$$\begin{aligned} \text{dim Ker } \phi_E''(\mu^*, x^*) &= N - 1 \qquad \textit{as } k = 2 \\ \text{dim Ker } \phi_E''(\mu^*, x^*) &= 2N - 3 \qquad \textit{as } k \geq 3, \end{aligned}$$

there exists $d_0 > 0$ such that for any $d \in (0, d_0)$, (P_ε) has, for ε small enough, a solution which blows up at k points $\xi_1, \ldots, \xi_k$ in Ω_d as ε goes to zero. Moreover, up to a central rotation, ξ/d goes to x^* as d and ε go to zero.

The case $k = 2$.

In view of Proposition 4.1, we just have to check, for proving Theorem 1.3, that $\dim \operatorname{Ker} \phi_E''(\mu^*, x^*) = N - 1$. For the sake of simplicity, we write $x = x_1^* = -x_2^*$, $r = r_0$ where $r_0 > 1$ solves $\Gamma_E'(r_0) = 0$, i.e.

$$\frac{1}{2^{N-1} r_0^N} = \frac{1}{(r_0^2 - 1)^{N-1}} + \frac{1}{(r_0^2 + 1)^{N-1}}. \tag{4.10}$$

Such an r_0 is unique. Indeed, derivating (2.5) while $\Gamma_E'(r) = 0$, we obtain

$$\Gamma_E''(r) = 2(N-2)\left[\frac{(N-2)r^2 + N}{(r^2-1)^N} + \frac{(N-2)r^2 + N}{(r^2+1)^N}\right] > 0$$

whence the unicity of the critical point in $(1, \infty)$. We also write $\mu = \mu^* = (-\Gamma_E(r_0))^{-1/2}$, i.e.

$$\mu = \left(\frac{1}{2^{N-2} r_0^{N-2}} - \frac{1}{(r_0^2-1)^{N-2}} - \frac{1}{(r_0^2+1)^{N-2}}\right)^{-1/2}. \tag{4.11}$$

Let $(e_1, e_2, f_1, \ldots, f_N, g_1, \ldots, g_N)$ be the dual basis of the $2N+2$ variables $\big(\mu_1, \mu_2, (x_1)_1, \ldots, (x_1)_N, (x_2)_1, \ldots, (x_2)_N\big)$. According to the definition of ϕ_E we compute

$$\phi_E''(\mu^*, x^*) = \begin{pmatrix}
A & B & C & \cdot & \cdots & \cdot & C & \cdot & \cdots & \cdot \\
B & A & -C & \cdot & \cdots & \cdot & -C & \cdot & \cdots & \cdot \\
C & -C & D & \cdot & \cdots & \cdot & E & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & F & \cdots & \cdot & \cdot & F & \cdots & \cdot \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\cdot & \cdot & \cdot & \cdot & \cdots & F & \cdot & \cdot & \cdots & F \\
C & -C & E & \cdot & \cdots & \cdot & D & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & F & \cdots & \cdot & \cdot & F & \cdots & \cdot \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\cdot & \cdot & \cdot & \cdot & \cdots & F & \cdot & \cdot & \cdots & F
\end{pmatrix} \tag{4.12}$$

where horizontal or vertical dots mean zeros, diagonal dots mean F, and, taking account of (4.10) and (4.11)

$$
\begin{aligned}
A &= R(x) - \frac{1}{\mu^2} = -\frac{1}{2^{N-2} r^{N-2}} + \frac{2}{(r^2-1)^{N-2}} + \frac{1}{(r^2+1)^{N-2}} \\
B &= -G(x,-x) = -\frac{1}{2^{N-2} r^{N-2}} + \frac{1}{(r^2+1)^{N-2}}
\end{aligned}
$$

$$
\begin{aligned}
C &= \mu \left(R'_{f_1}(x) - G'_{f_1}(x,-x) \right) = -\mu G'_{g_1}(x,-x) \\
&= \mu G'_{f_1}(x,-x) = -\mu \left(R'_{g_1}(-x) - G'_{g_1}(x,-x) \right) \\
&= -(N-2)\mu \frac{r}{(r^2-1)^{N-1}} \\
D &= \frac{\mu^2}{2} \left(R''_{f_1 f_1}(x) - 2G''_{f_1 f_1}(x,-x) \right) \\
&= \frac{\mu^2}{2} \left(R''_{g_1 g_1}(-x) - 2G''_{g_1 g_1}(x,-x) \right) \\
&= \frac{N-2}{2}\mu^2 \left[\frac{(3N-5)r^2 + (N+1)}{(r^2-1)^N} + \frac{(N-1)r^2 - (N-1)}{(r^2+1)^N} \right] \\
E &= -\mu^2 G''_{f_1 g_1}(x,-x) = -\mu^2 G''_{f_1 g_1}(x,-x) \\
&= \frac{N-2}{2}\mu^2 \left[\frac{N-1}{(r^2-1)^{N-1}} + \frac{(-N+3)r^2 + (N+1)}{(r^2+1)^N} \right] \\
F &= \frac{\mu^2}{2} \left(R''_{f_i f_i}(x) - 2G''_{f_i f_i}(x,-x) \right) = -\mu^2 G''_{f_i g_i}(x,-x) \\
&= -\mu^2 G''_{f_i g_i}(x,-x) = \frac{\mu^2}{2} \left(R''_{g_i g_i}(-x) - 2G''_{g_i g_i}(x,-x) \right) \\
&= -\frac{N-2}{2}\mu^2 \left[\frac{1}{(r^2-1)^{N-1}} + \frac{r^2-1}{(r^2+1)^N} \right], \quad 2 \le i \le N.
\end{aligned}
$$

In view of (2.3), we used

$$
\begin{aligned}
\frac{\partial R_E}{\partial \xi_i}(\xi) &= -2(N-2)\frac{\xi_i}{(|\xi|^2-1)^{N-1}} \\
\frac{\partial G_E}{\partial \xi_i}(\xi,\zeta) &= -(N-2) \left[\frac{\xi_i - \zeta_i}{|\xi-\zeta|^{N-1}} - \frac{|\zeta|^2 \xi_i - \zeta_i}{\left| |\zeta|\xi - \frac{\zeta}{|\zeta|} \right|^N} \right]
\end{aligned}
\tag{4.13}
$$

$$
\begin{aligned}
\frac{\partial R_E^2}{\partial \xi_i \partial \xi_j}(\xi) &= -2(N-2)\left[\frac{\delta_{ij}}{(|\xi|^2-1)^{N-1}} - 2(N-1)\frac{\xi_i\xi_j}{(|\xi|^2-1)^N}\right] \\
\frac{\partial^2 G_E}{\partial \xi_i \partial \xi_j}(\xi,\zeta) &= -(N-2)\left[\frac{\delta_{ij}}{|\xi-\zeta|^N} - N\frac{(\xi_i-\zeta_i)(\xi_j-\zeta_j)}{|\xi-\zeta|^{N+2}}\right. \\
&\qquad \left. - \frac{|\zeta|^2\delta_{ij}}{\left||\zeta|\xi - \frac{\zeta}{|\zeta|}\right|^N} + N\frac{(|\zeta|^2\xi_i-\zeta_i)(|\zeta|^2\xi_j-\zeta_j)}{\left||\zeta|\xi - \frac{\zeta}{|\zeta|}\right|^{N+2}}\right]
\end{aligned}
\tag{4.14}
$$

and

$$
\begin{aligned}
\frac{\partial^2 G_E}{\partial \xi_i \partial \zeta_j}(\xi,\zeta) =&(N-2)\left[\frac{\delta_{ij}}{|\xi-\zeta|^N} - N\frac{(\xi_i-\zeta_i)(\xi_j-\zeta_j)}{|\xi-\zeta|^{N+2}}\right. \\
&\qquad \left. + \frac{2\xi_i\zeta_j-\delta_{ij}}{\left||\zeta|\xi - \frac{\zeta}{|\zeta|}\right|^N} - N\frac{(|\zeta|^2\xi_i-\zeta_i)(|\zeta|^2\xi_j-\zeta_j)}{\left||\zeta|\xi - \frac{\zeta}{|\zeta|}\right|^{N+2}}\right]
\end{aligned}
\tag{4.15}
$$

Considering (4.12), we find the $2N$ following eigenvectors for $\phi_E''(\mu^*, x^*)$:

$$
\begin{aligned}
&\frac{1}{\sqrt{2}}(f_i - g_i),\ 2 \le i \le N, && \text{with eigenvalue } 0 \\
&\frac{1}{\sqrt{2}}(f_i + g_i),\ 2 \le i \le N, && \text{with eigenvalue } 2F \\
&\frac{1}{\sqrt{2}}(e_1 + e_2), && \text{with eigenvalue } A+B \\
&\frac{1}{\sqrt{2}}(f_1 - g_1), && \text{with eigenvalue } D-E
\end{aligned}
$$

and in the remaining orthonormal basis $\frac{1}{\sqrt{2}}(e_1 - e_2)$, $\frac{1}{\sqrt{2}}(f_1 + g_1)$, $\phi_E''(\mu^*, x^*)$ writes as

$$
\begin{pmatrix} A-B & 2C \\ 2C & D+E \end{pmatrix}.
$$

It is easily checked that $F < 0$, $A + B < 0$ (because of (4.10)), $D - E > 0$ and $(A - B)(D + E) - 4C^2 > 0$. Consequently

$$
\dim \operatorname{Ker} \phi_E''(\mu^*, x^*) = N - 1
$$

and Theorem 1.2 follows from Proposition 4.1.

<u>The case $k \ge 3$.</u>

For the sake of simplicity, we limit ourselves to the case $k = 3$. The other cases may be treated in the same way, with additional computations. In view of Proposition 4.1, we have to compute the dimension of the kernel of $\phi''_E(\mu^*, x^*)$, where (μ^*, x^*) is given by (4.9), where $r_0 > 1$ solves $\Gamma'_E(r_0) = 0$, i.e.

$$\frac{1}{3^{\frac{N-2}{2}} r_0^N} = \frac{1}{(r_0^2 - 1)^{N-1}} + \frac{2r_0^2 + 1}{(r_0^4 + r_0^2 + 1)^{\frac{N}{2}}}. \tag{4.16}$$

As previously, such an r_0 is unique. Indeed, derivating (2.5) we find, as $\Gamma'_E(r) = 0$

$$\Gamma''_E(r) = 2(N-2) \left[\frac{(N-2)r^2 + N}{(r^2 - 1)^N} + \frac{N(2r^2 + 1)(r^4 - 1)}{(r^4 + r^2 + 1)^{\frac{N+2}{2}}} \right] > 0$$

whence again the unicity of the critical point in $(1, \infty)$. We note also that, according to (2.5)

$$\Gamma'_E(2) < -(N-2) \left(\frac{1}{3^{N-1}} - \frac{1}{3^{\frac{N-2}{2}} 2^N} \right) < 0$$

implying that $r_0 > 2$. Furthermore $\mu^* = (-\Gamma_E(r_0))^{-1/2}$, i.e.

$$\mu^* = \left(\frac{2}{3^{\frac{N-2}{2}} r_0^{N-2}} - \frac{1}{(r_0^2 - 1)^{N-2}} - \frac{2}{(r_0^4 + r_0^2 + 1)^{\frac{N-2}{2}}} \right)^{-1/2}. \tag{4.17}$$

For the sake of simplicity, we write $x^* = (x_1, x_2, x_3)$, $r = r_0$ and $\overline{r} = (r_0^4 + r_0^2 + 1)^{\frac{1}{4}}$. Let $(e_1, e_2, e_3, f_1, \ldots, f_N, g_1, \ldots, g_N, h_1, \ldots, h_N)$ be the dual basis of the $3N+3$ variables $\big(\mu_1, \mu_2, \mu_3, (x_1)_1, \ldots, (x_1)_N, (x_2)_1, \ldots, (x_2)_N, (x_3)_1, \ldots, (x_3)_N\big)$. The subspaces $\mathcal{E}_i = \mathrm{Span}(f_i, g_i, h_i)$, $3 \le i \le N$, are stable for $\phi''_E(\mu^*, x^*)$, and

$$\phi''_E(\mu^*, x^*)\big|_{\mathcal{E}_i} = \alpha \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

with

$$\begin{aligned} \alpha &= \frac{\mu^{*2}}{2} \left(R''_{f_i f_i}(x_1) - 2G''_{f_i f_i}(x_1, x_2) - 2G''_{f_i f_i}(x_1, x_3) \right) \\ &= -\mu^{*2} G''_{f_i g_i}(x_1, x_2) = -\mu^{*2} G''_{f_i h_i}(x_1, x_3) \\ &= (N-2)\mu^{*2} \left[\frac{2}{3^{\frac{N}{2}} r^N} - \frac{1}{(r^2 - 1)^{N-1}} - \frac{2r^2}{\overline{r}^{2N}} \right] < 0 \end{aligned}$$

using (4.14), (4.15) and (4.16). The eigenvalues of $\phi''_E(\mu^*, x^*)\Big|_{\mathcal{E}_i}$ are 3α with multiplicity 1, and 0 with multiplicity 2 - whence $2(N-2)$ vectors in the kernel of $\phi''_E(\mu^*, x^*)$. Because of the symmetries of ϕ_E, we know three additional eigenvectors:

$$
\begin{aligned}
u &= \frac{1}{\sqrt{3}}(e_1 + e_2 + e_3), \text{ with eigenvalue } \Gamma_E(r) < 0; \\
v &= \frac{1}{\sqrt{3}}\left[f_1 + \left(-\frac{1}{2}g_1 + \frac{\sqrt{3}}{2}g_2 \right) + \left(-\frac{1}{2}h_1 - \frac{\sqrt{3}}{2}h_2 \right) \right], \\
&\quad \text{with eigenvalue } \tfrac{\mu^2}{2}\Gamma''_E(r) > 0; \\
w &= \frac{1}{\sqrt{3}}\left[f_2 + \left(-\frac{\sqrt{3}}{2}g_1 - \frac{1}{2}g_2 \right) + \left(\frac{\sqrt{3}}{2}h_1 - \frac{1}{2}h_2 \right) \right], \\
&\quad \text{with eigenvalue } 0.
\end{aligned}
$$

Therefore, Proposition 4.1 will provide us with a solution to (P_ε) blowing up at three points as ε goes to zero, if the restriction of $\phi''_E(\mu^*, x^*)$ to the stable $6-$dimensional orthogonal subspace $\mathcal{E}$ to the $\mathcal{E}_i$'s, $3 \le i \le N$, and to u, v, w, is nondegenerate. As a basis of $\mathcal{E}$, we can take

$$
\begin{aligned}
k_1 &= \frac{\sqrt{2}}{\sqrt{3}}\left(e_1 - \frac{1}{2}e_2 - \frac{1}{2}e_3 \right) \\
k_2 &= \frac{1}{\sqrt{2}}\left(e_2 - e_3 \right) \\
k_3 &= \frac{1}{\sqrt{3}}\left(f_1 + g_1 + h_1 \right) \\
k_4 &= \frac{1}{\sqrt{3}}\left(f_2 + g_2 + h_2 \right) \\
k_5 &= \frac{1}{\sqrt{3}}\left[f_1 + \left(-\frac{1}{2}g_1 - \frac{\sqrt{3}}{2}g_2 \right) + \left(-\frac{1}{2}h_1 + \frac{\sqrt{3}}{2}h_2 \right) \right] \\
k_6 &= \frac{1}{\sqrt{3}}\left[f_2 + \left(\frac{\sqrt{3}}{2}g_1 - \frac{1}{2}g_2 \right) + \left(-\frac{\sqrt{3}}{2}h_1 - \frac{1}{2}h_2 \right) \right].
\end{aligned}
$$

In this basis we have

$$\phi''_E(\mu^*, x^*)\Big|_{\mathcal{E}} = \begin{pmatrix} A & 0 & B & 0 & C & 0 \\ 0 & A & 0 & B & 0 & -C \\ B & 0 & D & 0 & E & 0 \\ 0 & B & 0 & D & 0 & -E \\ C & 0 & E & 0 & F & 0 \\ 0 & -C & 0 & -E & 0 & F \end{pmatrix} \tag{4.18}$$

with

$$\begin{aligned} A &= \frac{2}{(r^2-1)^{N-2}} - \frac{1}{3^{\frac{N-2}{2}} r^{N-2}} + \frac{1}{\overline{r}^{2(N-2)}} \\ B &= -\frac{N-2}{\sqrt{2}}\mu^* r\left[\frac{2}{(r^2-1)^{N-1}} + \frac{r^2-1}{\overline{r}^{2N}}\right] \\ C &= \frac{N-2}{\sqrt{2}}\mu^* r\left[\frac{1}{(r^2-1)^{N-1}} - \frac{r^2-1}{\overline{r}^{2N}}\right] \\ D &= (N-2)\mu^{*2}\left[\frac{(N-2)r^2+1}{(r^2-1)^N} - \frac{r^2-2}{\overline{r}^{2N}} + \frac{Nr^2(r^2-1)^2}{2\overline{r}^{2(N+2)}}\right] \\ E &= (N-2)\mu^{*2}\left[\frac{(N-1)r^2}{(r^2-1)^N} + \frac{r^2}{\overline{r}^{2N}} + \frac{Nr^2(r^4+r^2-2)}{2\overline{r}^{2(N+2)}}\right] \\ F &= \frac{N-2}{2}\mu^{*2}\left[\frac{(N-2)r^2+N}{(r^2-1)^N} - \frac{2(N-1)r^2+N}{2\overline{r}^{2N}} + \frac{Nr^2(r^2+2)^2}{2\overline{r}^{2(N+2)}}\right]. \end{aligned} \tag{4.19}$$

For example

$$\begin{aligned} A &= k_1^t.\phi''_E(\mu^*, x^*).k_1 \\ &= \frac{2}{3}\left[\Big(\frac{\partial^2\phi''_E}{\partial\mu_1^2} - \frac{1}{2}\frac{\partial^2\phi''_E}{\partial\mu_1\partial\mu_2} - \frac{1}{2}\frac{\partial^2\phi''_E}{\partial\mu_1\partial\mu_3}\Big) - \frac{1}{2}\Big(\frac{\partial^2\phi''_E}{\partial\mu_1\partial\mu_2} - \frac{1}{2}\frac{\partial^2\phi''_E}{\partial\mu_2^2}\right. \\ &\qquad \left. -\frac{1}{2}\frac{\partial^2\phi''_E}{\partial\mu_2\partial\mu_3}\Big) - \frac{1}{2}\Big(\frac{\partial^2\phi''_E}{\partial\mu_1\partial\mu_3} - \frac{1}{2}\frac{\partial^2\phi''_E}{\partial\mu_2\partial\mu_3} - \frac{1}{2}\frac{\partial^2\phi''_E}{\partial\mu_3^2}\Big)\right]. \end{aligned}$$

We have

$$\begin{aligned} \frac{\partial^2\phi''_E}{\partial\mu_1^2} = \frac{\partial^2\phi''_E}{\partial\mu_2^2} = \frac{\partial^2\phi''_E}{\partial\mu_3^2} = R(x_1) - \frac{1}{\mu^{*2}} \\ \frac{\partial^2\phi''_E}{\partial\mu_1\partial\mu_2} = \frac{\partial^2\phi''_E}{\partial\mu_1\partial\mu_3} = \frac{\partial^2\phi''_E}{\partial\mu_2\partial\mu_3} = -G(x_1, x_2) \end{aligned}$$

whence the announced result, observing (2.3) and (4.9). The other quantities are obtained in the same way, using also (4.13), (4.14), (4.15) and (4.16).

All these quantities are strictly positive, except B which is strictly negative. Actually, $B < 0$ and $E > 0$ are trivial. $C > 0$, $D > 0$ and $F > 0$ follow directly from the fact that, according to the definition of $\overline{r}$, $r^2 - 1 < \overline{r}^2$. Lastly, we deduce from (4.16)

$$A = \frac{r^2 - 2}{(r^2 - 1)^{N-1}} - \frac{r^4 - 1}{\overline{r}^{2N}}.$$

Therefore, $A > 0$ is equivalent to

$$(r^2 - 2)\overline{r}^{2N} > (r^2 + 1)(r^2 - 1)^N$$

or

$$\Big(\frac{r^4 + r^2 + 1}{r^4 - 2r^2 + 1}\Big)^{\frac{N}{2}} > \frac{r^2 + 1}{r^2 - 2} = 1 + \frac{3}{r^2 - 2}. \tag{4.20}$$

As $(1 + X)^\alpha > 1 + \alpha X$ for $X > 0$ and $\alpha > 1$

$$\Big(\frac{r^4 + r^2 + 1}{r^4 - 2r^2 + 1}\Big)^{\frac{N}{2}} > 1 + \frac{N}{2}\frac{3r^2}{r^4 - 2r^2 + 1}$$

and, in view of (4.20), $A > 0$ follows from the fact that (as $r > 2$)

$$\frac{N}{2}\frac{1}{r^2 - 2 + \frac{1}{r^2}} > \frac{1}{r^2 - 2} \quad \text{or} \quad (N - 2)(r^2 - 2) > \frac{2}{r^2}.$$

We note now that through suitable exchanges between lines and columns, we deduce from (4.18)

$$\det \phi''_E(\mu^*, x^*)\Big|_{\mathcal{E}} = \begin{vmatrix} A & B & C & 0 & 0 & 0 \\ B & D & E & 0 & 0 & 0 \\ C & E & -F & 0 & 0 & 0 \\ 0 & 0 & 0 & A & B & -C \\ 0 & 0 & 0 & B & D & -E \\ 0 & 0 & 0 & -C & -E & F \end{vmatrix}$$

whence

$$\det \phi''_E(\mu^*, x^*)\Big|_{\mathcal{E}} = \Big[A(DF - E^2) - (FB^2 + DC^2 - 2EBC)\Big]^2$$

where A, B, C, D, E, F are defined by (4.16) and (4.19). Then, the nondegeneracy of $\phi''_E(\mu^*, x^*)$ restricted to $\mathcal{E}$ is equivalent to

$$A(DF - E^2) \neq FB^2 + DC^2 - 2EBC.$$

As C, D, E, F are strictly positive and B is strictly negative, the right hand side is strictly positive. As we have also $A > 0$, the inequality $DF - E^2 < 0$ is sufficient to conclude. Considering that D, F are strictly positive, the result will follow from the two inequalities $D < E$ and $F < E$. In view of (4.19), $D < E$ is obvious and $F < E$ is equivalent to

$$\frac{N}{(r^2-1)^{N-1}} + \frac{2Nr^2+N}{2\overline{r}^{2N}} + \frac{Nr^2(r^4-2r^2-8)}{2\overline{r}^{2(N+2)}} > 0$$

As $r^2 - 1 < \overline{r}^2$, this inequality will be satisfied if

$$(4r^2-1)\overline{r}^4 + r^2(r^4-2r^2-8) = 5r^6 + r^4 - 5r^2 - 1 > 0$$

which is true since $r > 1$.

Collecting the previous informations, we know that the dimension of the kernel of $\phi''_E(\mu^*, x^*)$ is exactly $2N-3$. Then, in view of Proposition 4.1 we can state:

Theorem 4.4. *There exists $d_0 > 0$ such that for any $d \in (0, d_0)$, (P_ε) has, for ε small enough, a solution which blows up at three points ξ_1, ξ_2, ξ_3 of Ω_d as ε goes to zero. Moreover, $|\xi_i| \sim r_0 d$ and there is a rotation $\mathcal{R}$ in $(\mathbb{R}^N)^3$, $\mathcal{R}^3 = \mathrm{id}$, such that $\xi_2 = \mathcal{R}\xi_1 + o(d)$ and $\xi_3 = \mathcal{R}^2\xi_1 + o(d)$.*

Acknoledgements. The authors are very grateful to the referee for pointing out several inaccuracies in the first version of the paper, the correction of which allowed to improve the formulation of the results.

References

[1] T. Aubin, *Problemes isoperimetriques et espaces de Sobolev* , J. Diff. Geom. **11** (1976), 573–598.

[2] A. Bahri, Critical point at infinity in some variational problems, *Pitman Research Notes Math.* **182** Longman House, Harlow (1989).

[3] A. Bahri, J.M. Coron, *On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain*, Comm. Pure Appl. Math. **41** (1988), 253–294.

[4] A. Bahri, Y.Li, O. Rey, *On a variational problem with lack of compactness: the topological effect of the critical points at infinity*, Calc. of Variat. **3** (1995), 67–93.

[5] M. Ben Ayed, K. El Mehdi, M. Grossi, O. Rey, *A nonexistence result of single peaked solutions to a supercritical nonlinear problem*, Comm. Contemp. Math. (to appear).

[6] L. Caffarelli, B. Gidas, J. Spruck, *Asymptotic symmetry and local behaviour of semilinear elliptic equations with critical Sobolev growth,* Comm. Pure Appl. Math. **42** (1989), 271–297.

[7] J.M. Coron, *Topologie et cas limite des injections de Sobolev*, C. R. Math. Acad. Sci. Paris **299** (1984), 209–212.

[8] E.N. Dancer, *A note on an equation with critical exponent,* Bull. London Math. Soc. **20** (1988), 600–602.

[9] M. Del Pino, P. Felmer, M. Musso, *Two-bubble solutions in the super-critical Bahri-Coron's problem*, Calc. Variat. PDE **16** (2003), 113–145.

[10] M. Del Pino, P. Felmer, M. Musso, *Multi-bubble solutions for slightly super-critical elliptic problems in domains with symmetries*, Bull. London Math. Soc. **35** (2003), 513–521.

[11] W.Y. Ding, *Positive solutions of $\Delta u + u^{\frac{N+2}{N-2}}$ on contractible domains,* J. Partial Diff. Equat. **2** (1989), 83–88.

[12] J. Kazdan, F. Warner, *Remarks on some quasilinear elliptic equations*, Comm. Pure Appl. Math. **28** (1975), 567–597.

[13] S. Khenissy, O. Rey, *A criterion for existence of solutions to the supercritical Bahri-Coron's problem*, Houston Journ. Math. **30** (2004), 587–613.

[14] R. Molle, D. Passaseo, *Positive solutions for slightly super-critical elliptic equations in contractible domains*, C. R. Math. Acad. Sci. Paris **335** (2002), 459–462.

[15] R. Molle, D. Passaseo, *On the existence of positive solutions of slightly supercritical elliptic equations*, Adv. Nonlinear Stud. **3** (2003), 301–326.

[16] R. Molle, A. Pistoia, *Concentration phenomena in elliptic problems with critical and supercritical growth,* Advances in Diff. Equat. **8** (2003), 547–570.

[17] M. Musso, A. Pistoia, *Multispike solutions for a nonlinear elliptic problem involving critical Sobolev exponent*, Indiana Univ. Math. J. **51** (2002), 541–579.

[18] D. Passaseo, *Multiplicity of positive solutions of nonlinear elliptic equations with critical Sobolev exponent in some contractible domains*, Manuscripta Math. **65** (1989), 147–165.

[19] D. Passaseo, *New nonexistence results for elliptic equations with supercritical nonlinearity*, Differential Int. Equat. **8** (1995), 577–586.

[20] S.I. Pohozaev, *On the eigenfunctions of the equation* $\Delta u + \lambda f(u) = 0$, Soviet. Math. Dokl. **6** (1965), 1408–1411.

[21] O. Rey, *The role of the Green's function in a nonlinear elliptic equation involving the critical Sobolev exponent*, J. Funct. Anal. **89** (1990), 1–52.

[22] O. Rey, *Sur un probéme variationnel non compact: l'effet de petits trous dans le domain*, C. R. Math. Acad. Sci. Paris **308** (1989), 349–352.

[23] G. Talenti, *Best constants in Sobolev inequality*, Ann. Mat. Pura Appl. **110** (1976), 353–372.